

# Fermi 1D quantum gas: Luttinger liquid approach and spin-charge separation

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We discuss the properties of quasi-1D quantum gases of fermionic atoms using the Luttinger liquid theory, including the presence of an optical lattice and of a longitudinal trapping potential. We analyze in particular the nature and manifestations of spin-charge separation, where in the case of atoms “spin” and “charge” refers to two internal atomic states and the atomic mass density, respectively.

## I. INTRODUCTION

The recent experimental possibilities to trap ultra-cold quantum gases in quasi-1D magnetic and optical waveguide aroused a renewed interest in quantum liquids with reduced dimensionality and their peculiar properties. In this context, most of the recent theoretical and experimental work has focused on quasi-1D Bose gases and Bose-Einstein condensates [1,2]. In particular, the equilibrium properties have been already discussed in several papers (see Ref. [3] and references therein), the phase fluctuations of the classical field (order parameter) describing a quasi-1D condensate have been predicted in Ref. [4], and experimental evidence was reported in Ref. [5]. In Ref. [6] the collective excitations for 1D Bose gases have been calculated for different configurations, ranging from the 1D mean-field to the Tonks regime.

At the same time also the possibility to include an optical lattice has become available, opening new perspectives in the study of coherence phenomena [7]. For instance Bloch oscillation have been seen in Ref. [8]. In Ref. [9], it is claimed that a direct measurement of the critical Josephson current has been obtained, and very recently the intriguing quantum phase transition between superfluid and Mott-insulator, predicted some years ago in Ref. [10], was observed for a Bose gas loaded in a 3D optical lattice in Ref. [11]. Also the dynamics of a trapped Bose-Einstein condensate in the presence of 2D and 1D optical lattice has been calculated in Ref. [12].

On the other hand remarkable progress in trapping and cooling Fermi gases, has been made to reach the degenerate regime [13]. On the theoretical side different aspects of (harmonically) trapped Fermi gases have been explored, often in analogy with what has been done for Bose gases [14–16]. Experimentally, measurements of the collective modes of degenerate Fermi gases have been reported (see Ref. [17] and reference therein).

Fermi cold-gases, besides having an interest *per se*, can represent a useful tool to implement solid state systems and to test fundamental theories, often grown and de-

veloped in a different area. Trapped gases are cleaner and more flexible systems than the solid state counterpart, where the experimentally most relevant 1D systems are semiconductor quantum wires and nanotubes. In addition, it is also possible to use magnetic fields to largely modify the scattering properties of atoms in atomic clouds [18].

Returning to 1D systems, what is interesting is that due to the reduced dimensionality the interaction, although small, strongly modifies the properties of the system. Low dimensions amplify the role of quantum fluctuations and enhance correlations. Both the ground state and the excitations exhibit strong correlation effects and possess a number of exotic properties, e.g., fractional statistics (see Ref. [19–21] and references therein). Another fundamental interest in 1D systems originates from the fact that there is a fair number of exactly solvable models, allowing the research of non-perturbative effects [22]. In particular Fermi-Landau theory which describes 3D interacting fermions is not longer applicable (e.g., see [20]). Instead, in a quite wide range of parameters, 1D quantum liquids can be described using the Tomonaga-Luttinger model [23], and in this case they are named *Luttinger liquids*, in contrast to (normal) Fermi liquids. The term Luttinger liquid was coined by Haldane [19] to describe the universal low-energy properties of gapless 1D quantum systems. The Luttinger liquid language has been already applied to study the properties of (quasi)-1D Bose gases [24,25]. The goal of this paper is to study trapped ultra-cold Fermi gases in quasi-1D geometry using Luttinger theory and to propose an experimental way to see a typical Luttinger liquid behaviour, the so-called *spin-charge separation*. Indeed one of the hallmarks of a spin-1/2 Luttinger liquid involves the complete separation in the dynamics of spin and charge degrees of freedom. Both branches, which exhaust (in a infinite system) the spectrum, are sound-like with different propagation velocities. This phenomena has never seen in a clear way in actual condensed matter systems (see e.g. [26]).

The paper, which can be considered a self-consistent, extended version of Ref. [27], is structured as follows. In SEC. II we introduce the models. Using standard argument, we write down a bosonized Hamiltonian to describe the low-energy excitations of the systems (SEC. III). We used the available exact solutions to relate the so-called Luttinger parameters to the microscopic parameters of the system for small and large interaction (SEC. III). In SEC. IV we shall propose different experimental ways to see clearly spin-charge separation in our systems, and in

SEC. V we study the life time of the excitations. In SEC. VI we propose an implemetation of spinless Luttinger liquid using strongly interacting bosons in an optical lattice.

## II. PHYSICAL MODEL

We consider a dilute gas of fermionic atoms of mass  $m$  with two ground states,  $|\sigma = \uparrow, \downarrow\rangle$ , representing a spin-1/2, e.g. the states  $|F = 1/2, M_F = \pm 1/2\rangle$  of  $^6\text{Li}$  [28]. In the following we assume the number of atom in each level is the same:  $N_\sigma = N/2$ . The atoms are confined in a very elongated trap along the  $x$ -direction and cooled below the Fermi temperature  $T_F$ . The transverse confinement is considered to be harmonic with frequency  $\omega_\perp$ , strong enough so that the transverse degrees of freedom frozen. In order to have a (quasi) 1D system the tight transverse trapping must exceed the characteristic energy scale of the longitudinal motion. Due to the quantum degeneracy the longitudinal motion has all the energy levels up to the Fermi-energy  $\epsilon_F \sim k_B T_F$  occupied, thus the requirement is  $\epsilon_F \ll \hbar\omega_\perp$ .

At low temperature, due to the Pauli exclusion principle, only inter-species s-wave collisions are allowed. All the relevant interactions are thus characterized by a single parameter, the inter-component scattering length  $a$ . Introducing the harmonic oscillator transverse length  $l_\perp = \sqrt{\hbar/m\omega_\perp}$ , the effective 1D interaction can be represented as a zero-range potential with strength  $g = 2\pi\hbar^2 a/m l_\perp^2$ , when  $a \ll l_\perp$  [2]. Thus the system can be described by the following 1D Hamiltonian

$$H = \sum_\sigma \int dx \psi_\sigma^\dagger(x) \left[ \frac{-\hbar^2}{2m} \partial_x^2 + V_{\text{ext},\sigma}(x) \right] \psi_\sigma(x) + g \int dx \psi_\uparrow^\dagger(x) \psi_\downarrow^\dagger(x) \psi_\uparrow(x) \psi_\downarrow(x), \quad (1)$$

where  $\psi_\sigma$  is the 1D field operator for atoms in the state  $\sigma$ . The external potential  $V_{\text{ext},\sigma}$ , includes both the longitudinal confinement  $V_{L,\sigma}$  and a possible superimposed optical lattice  $V_{\text{opt},\sigma}$ . The longitudinal confinement is either a box of length  $L$  (we will also refer to this as the homogeneous case), or an harmonic potential (inhomogeneous gas) with frequency  $\omega_x$ , i.e.  $V_{L,\sigma} = 1/2 m \omega_x^2 x^2$ . The confinement potential can act differently on the two species, and this feature can be used in an experiment, as we will show below, to test the “spin-charge separation”. The simplest way to create a 1D periodic potential for neutral atoms is by superimposing two linearly polarized counter-propagating laser beams. The created conservative potential is  $V_{\text{opt},\sigma} = V_0 \sin^2(kx)$ , FIG. (1), where the lattice period  $d = \pi/k = \lambda/2$  is fixed by the wavelength,  $\lambda$ , of the laser light (see, e.g., [29]). We can expand the field operators in the first band Wannier basis

$$\psi_\sigma(x) = \sum_i w(x - x_i) c_{i,\sigma}, \quad (2)$$

where  $w(x - x_i)$  is the Wannier function centered at the  $i$ -th site and  $c_{i,\sigma}$  the annihilation operators for a fermions in the  $i$ -th site with spin  $\sigma$ . Plugging the previous expression into Eq. (1) and retaining at most nearest neighbor terms for the kinetic energy part we reduce the starting Hamiltonian to the Fermi-Hubbard Hamiltonian

$$H_{FH} = -J \sum_{i,\sigma} (c_{i+1,\sigma}^\dagger c_{i,\sigma} + h.c.) + \sum_{i,\sigma} \epsilon_{i,\sigma} n_{i,\sigma} + U \sum_i n_{i,\uparrow} n_{i,\downarrow}, \quad (3)$$

where operator  $n_{i,\sigma} = c_{i,\sigma}^\dagger c_{i,\sigma}$  counts the number of  $\sigma$ -atoms in the site  $i$ . The parameter  $U$  correspond to the strength of the on site inter-species interaction depending on  $g$ ,  $J$  is the hopping term between adjacent sites and  $\epsilon_{i,\sigma} \simeq V_{L,\sigma}(x_i)$  describes an energy offset of each lattice site. Note that for  $d \rightarrow 0$  one recovers a continuum model.

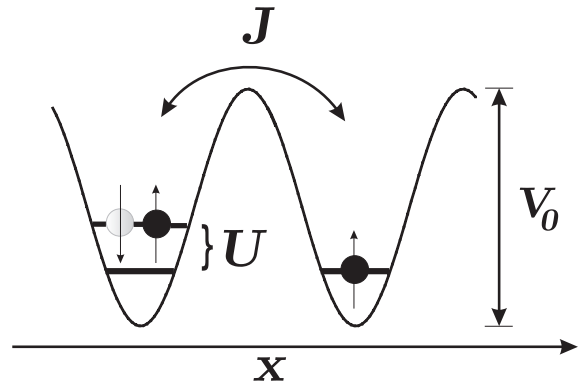


FIG. 1. Schematic representation of fermions in an optical lattice. The parameter  $J$  is the amplitude to jump from one site to the adjacent one.  $U$  is the on-site energy due to the interaction between atoms in different internal levels.

## III. LUTTINGER LIQUID APPROACH

In what follows we will often use the words charge and spin though in our system we do not have, strictly speaking, charge nor spin. We used them to maintain the jargon of the Luttinger physics. We identify the charge with the mass and the spin with the two atomic internal levels. Thus when we talk about charge density fluctuations we mean density fluctuations, when we talk about spin density fluctuations we mean relative density fluctuations, the so called composite excitations.

### A. Homogeneous case

In this subsection we will review briefly what it means to describe the low energy properties of the system using the so-called Tomonaga-Luttinger model.

We consider the homogeneous case, i.e.,  $V_{L,\sigma} = 0$ , and we will study later the inhomogeneous case. The standard way to represent the Eq. (1) for a gas in a box of length  $L$  with periodic boundary condition, without the underlying optical lattice, in the momentum representation is

$$H = \sum_{k,\sigma} \varepsilon_k c_{k,\sigma}^\dagger c_{k,\sigma} + \frac{g}{L} \sum_{k_1,k_2,q} c_{k_1,\uparrow}^\dagger c_{k_1-q,\uparrow} c_{k_2,\downarrow}^\dagger c_{k_2+q,\downarrow}, \quad (4)$$

where  $\varepsilon_k = \hbar^2 k^2 / 2m - \mu$ .

By means of renormalization group (RG), it is possible to show that the low-energy properties of the system can be described using an exact solvable model, the Tomonaga-Luttinger model [19–21,23]. First of all in a 1D system the Fermi “surface” is composed by just two points,  $\pm k_F$ . In the non-interacting case one has  $\pm k_F = \pm \pi \hbar n / 2$ , where  $n = (N_\uparrow + N_\downarrow) / L$  is the total density of the gas. Let us, then, introduce the left- (around  $-k_F$ ) and right- (around  $+k_F$ ) moving fermion annihilation (creation) operators  $c_{r,\sigma}^{(\dagger)}(k)$ , where  $r = R, L$ , and the respective densities  $\rho_{r,s}(q) = \sum_k c_{r,\sigma}^\dagger(k+q) c_{r,\sigma}(k)$ . In the model the four particle species ( $R \uparrow, R \downarrow, L \uparrow, L \downarrow$ ) have unbounded free dispersion relation  $\epsilon_{r,q} = a_r \hbar v_F q$ , where  $v_F = \hbar \pi n / 2m$  is the Fermi velocity and  $a_{R(L)} = +(-)$ ; all the states up to the Fermi energy, in the same spirit of the Dirac sea, are occupied. Thus one has two branches of particles with boundless energy and momentum and the densities obey Bose type commutation relation [20]. The model Hamiltonian written in the notation of the so-called “ $g$ -ology” (see for instance [30]), in real space representation, takes the form

$$\begin{aligned} H_{LM} = & \sum_\sigma \int dx \pi \hbar v_F (\rho_{R,\sigma}(x)^2 + \rho_{L,\sigma}(x)^2) \\ & + g_{2\perp} \sum_\sigma \int dx \rho_{R,\sigma}(x) \rho_{L,-\sigma}(x) \\ & + g_{4\perp} \sum_{r,\sigma} \int dx \rho_{r,\sigma}(x) \rho_{r,-\sigma}(x) \\ & + g_{1\perp} \sum_{r,\sigma} \int dx \psi_{L,\sigma}^\dagger \psi_{R,\sigma} \psi_{R,-\sigma}^\dagger \psi_{L,-\sigma}. \end{aligned} \quad (5)$$

Specific names are given to the parameters which identify the strength of a particular scattering process. For instance  $g_{2\perp}$  is the strength of the forward scattering between particles belonging to different branches and with different spin state,  $(R\sigma, L-\sigma) \rightarrow (R\sigma, L-\sigma)$ , while  $g_{1\perp}$  is the strength of the backward scattering  $(R\sigma, L-\sigma) \rightarrow (L\sigma, R-\sigma)$ . In a complete model also “longitudinal” scattering  $(r\sigma, r'\sigma) \rightarrow (r\sigma, r'\sigma)$  could be present. In our case, where only inter-species collisions are allowed, we have  $g_{2\parallel} = g_{4\parallel} = 0$ .

In the sense of RG in the long-wavelength limit, it is often the case that  $g_{1\perp}$  is renormalized to zero [20,31–33].

This is true in our case. The final step to solve the problem is to introduce four new boson fields  $\phi_\nu$ ,  $\Pi_\nu$ , with  $\nu = c, s$ , such that

$$\rho_{r,\sigma}(x) = \frac{1}{\sqrt{8\pi}} [\partial_x \phi_c - a_r \Pi_c + \sigma (\partial_x \phi_s - a_r \Pi_s)]. \quad (6)$$

The bosonic fields are related to the fluctuations of the total density  $\rho_c = \partial_x \phi_c / \sqrt{\pi}$ , of the spin-density  $\rho_s = \partial_x \phi_s / \sqrt{\pi}$ , of the current density  $j_c = -\Pi_c / \sqrt{\pi}$  and of the spin current density  $j_s = -\Pi_s / \sqrt{\pi}$ . The fields  $\phi$  and  $\Pi$  are conjugated and they satisfies the bosonic commutation relation  $[\phi_\nu(x), \phi_{\nu'}(x')] = [\Pi_\nu(x), \Pi_{\nu'}(x')] = 0$  and  $[\phi_\nu(x), \Pi_{\nu'}(x')] = i \delta_{\nu,\nu'} \delta(x-x')$ . Substituting Eq. (6) in Eq. (5) one obtains the bosonized Hamiltonian [20]

$$H = \sum_{\nu=c,s} \frac{u_\nu}{2} \int dx \left[ K_\nu \Pi_\nu^2 + \frac{1}{K_\nu} (\partial_x \phi_\nu)^2 \right]. \quad (7)$$

We are left with the Hamiltonian of two independent elastic strings with the eigenmodes corresponding to the collective density and spin-density fluctuation of the fermion liquid. The parameters  $u_\nu$ ’s are the “sound” velocities, while the  $K_\nu$ ’s are related to the low energy behaviour of the correlation functions. In the following we will call them the Luttinger parameters. They completely characterize the low energy physics.

For excitations which do not change neither the number of particles and the currents, the bosonic fields can be written in terms of boson creation and annihilation operators,  $b_{\nu,k}^\dagger$  and  $b_{\nu,k}$ ,  $\nu = c, s$

$$\phi_\nu(x) = \sqrt{\frac{K_\nu}{2L}} \sum_k \frac{1}{\sqrt{|k|}} \left( b_{\nu,k} e^{ikx} + b_{\nu,k}^\dagger e^{-ikx} \right), \quad (8)$$

$$\Pi_\nu(x) = i \sqrt{\frac{1}{2LK_\nu}} \sum_k \sqrt{|k|} \left( b_{\nu,k} e^{ikx} - b_{\nu,k}^\dagger e^{-ikx} \right). \quad (9)$$

It is moreover possible to express the original fermionic fields in terms of the new boson fields by the so-called bosonization identity (see for instance [20,31,34,22])

$$\psi_{r,\sigma} = \lim_{\alpha \rightarrow 0} \frac{F_{r,\sigma}}{(2\pi\alpha)^{1/2}} e^{-a_r i 2\sqrt{\pi} \phi_{R(L),\sigma}}. \quad (10)$$

The fields  $\phi_{r,\sigma}$  are recovered from Eq. (6). The (hermitian) operators  $F_{r,\sigma}$  are called Klein factors and obey the Clifford algebra  $\{F_{r,\sigma}, F_{r',\sigma'}\} = 2\delta_{r,r'} \delta_{\sigma,\sigma'}$ .

The same effective Hamiltonian, Eq. (7), can be obtained starting from the Hubbard Hamiltonian Eq. (3). In the optical lattice case, besides the back-scattering term, also an umklapp contribution could in principle be present, but it does not contribute away from half-filling.

### 1. Luttinger parameters

For exactly solvable models like the 1D lattice model, the Luttinger parameters can be directly expressed in

terms of the microscopic parameters of the theory [35]. In a spin rotationally invariant Fermi-gas the quantity  $K_s = 1$ , so that the only independent parameters are  $K_\rho$ , and  $v_{s,\rho}$  [19,20].

One of the more peculiar consequences of the Hamiltonian Eq. (7) is a complete separation of the dynamics of the spin and charge degrees of freedom. In general one has  $u_c \neq u_s$  and thus the density (or charge) and spin oscillations propagate with different velocities. Only for a non-interacting gas, or for some accident, the two velocities are the same and equal to the Fermi velocity,  $u_\nu = v_F$ . This phenomenon is known as *spin-charge separation*. Our aim is eventually to propose a way to see in a clear way such effect.

We should mention that if  $g_{1\perp}$  renormalizes to a finite value (in particular in a spin rotationally invariant system with bare  $g_{1\perp} < 0$ , the parameter gets relevant by means of RG), one can have a gap in the spin sector, the charge excitations remaining massless [20,31].

In what follows we shall give some analytical results for the Luttinger parameters of systems described by the Hamiltonian Eq. (1) for a box and a harmonically trapping potential, with and without the underlying optical lattice.

For weak interaction, we define the small parameter  $\xi = g/\pi\hbar v_F$ . We get, up to the first order in  $\xi$ ,  $u_c = v_F(1 + \xi/2)$ ,  $u_s = v_F(1 - \xi/2)$ ,  $K_c = 1 - \xi/2$ . The same results are valid in the presence of optical lattice, the only difference being in the definition of  $\xi = Ud/\pi\hbar v_F$ , with Fermi velocity  $v_F = 2Jd\sin(\pi nd/2)/\hbar$  [32].

When  $V_{L,\sigma} = 0$ , i.e. in the homogeneous case, the models described by the Eq. (1) and Eq. (3), are exactly solvable [36,37]. The solutions confirm that the low energy spectrum is Luttinger-like and so, at least in principle, it is possible to extract the Luttinger parameters for any interaction strength. The exact solution [36,37] is suitable to study the strong interacting limit, i.e.,  $\xi \gg 1$ . The parameters can be found using perturbation theory around  $\xi \rightarrow \infty$ . Without the lattice one has:  $u_s = 2\hbar\pi n/3m\xi$ ,  $K_c = 1/2(1 + 8\ln(2)/\pi^2\xi)$  and  $u_c = \pi\hbar n/m(1 - 8\ln(2)/\pi^2\xi)$ , where  $n = N/L$  is the density of the gas. When the lattice is present one obtains

$$u_s = \frac{Jd}{\hbar\sin(\pi nd/2)\xi} \left( 1 - \frac{\sin(2\pi nd)}{2\pi nd} \right), \quad (11)$$

$$K_c = \frac{1}{2} \left( 1 + \frac{4\ln(2)}{\pi^2\xi} \frac{\sin(\pi nd)}{\sin(\pi nd/2)} \right), \quad (12)$$

$$u_c = \frac{2Jd\sin(\pi nd)}{\hbar} \left( 1 - \frac{4\ln(2)nd\cos(\pi nd)}{\pi\sin(\pi nd/2)\xi} \right). \quad (13)$$

Some remarks can be made. First of all we should mention that the results without the lattice can be recovered from the previous expression expanding in the first (non-zero) order in low density  $n$  and substituting  $J \rightarrow \hbar^2/2md^2$ , the new density being  $n = N/N_w d \rightarrow N/L$ , with  $N_w$  the number of sites. Indeed from the exact solution one gets that, in the limit of low density, the lattice

case resembles the continuous case with a renormalized mass  $m^* = (2Jd^2/\hbar^2)^{-1}$ . Secondly, note that the results (put  $\xi \rightarrow \infty$ ) confirm the intuition that when the repulsion between the atoms of the two different species is very strong some properties of the gas are similar to those of an ideal single component gas of indistinguishable particles. In particular the Fermi velocity is  $v_F = \pi\hbar n/m$  and the spin velocity goes to zero. Intuitively one can say that, somehow, the infinite (hard-core or on-site) repulsion plays the role of an effective Pauli principle for atoms in different internal state (i.e., different spin). The same is true for instance for strongly interacting bosons in 1D, where there exist an exact mapping between hard core bosons and free fermions [38]. Note, however, that in our case the asymptotic value  $K_c = 1/2$  [35], while is  $K_c = 1$  for a non-interacting gas. It is well known that at zero temperature the Green's function of a Luttinger liquids present a peculiar power-law decay with non-universal,  $K$ -dependent exponents. At finite temperature the behaviour is still  $K$ -dependent [20]. This means that physical properties like compressibility, density of states, momentum distribution clearly reveal the interacting nature of the gas.

## 2. Inhomogeneous case: Local Density Approximation

The inhomogeneous case will be studied using a local density approximation (LDA). In order to apply LDA to our trapped gas we should assume that the size  $R$  of the gas sample is much larger than the interparticle separation, i.e.  $R \gg k_F^{-1}$ . Such condition is consistent with having a large number of particle,  $N \gg 1$ . The ground state of the system can be characterized using the Thomas-Fermi equilibrium condition:

$$\frac{dE_0(n)}{dn} = \mu - V_{L,\sigma}(x). \quad (14)$$

Where  $E_0(n)$  is the ground state internal energy of the gas per unit length,  $n$  the total density and  $\mu$  is the chemical potential. The longitudinal potential in this case is  $V_{L,\sigma}(x) = 1/2m\omega_x^2 x^2$ . The previous equation is just the expression of the fact that the energy cost of adding a particle to the system equals the chemical potential corrected by the local value of the external potential. In the Luttinger picture we assume that the spatial variation of  $K_\nu$  and  $u_\nu$  originate only from the spatial dependence of the gas density, i.e.,  $K_\nu[n] \rightarrow K_\nu[n(x)]$  and  $u_\nu[n] \rightarrow u_\nu[n(x)]$ .

The equation of motion for the fields conjugate fields representing the low energy excitations of the inhomogeneous gas are obviously given by  $i\dot{\phi}_\nu = [H, \phi_\nu]$  and  $i\dot{\Pi}_\nu = [H, \Pi_\nu]$ , where  $H$  is the Hamiltonian Eq. (7) with spatial dependent Luttinger parameters. Using the canonical commutation relation one can easily find:

$$\dot{\phi}_\nu = K_\nu(x)u_\nu(x)\Pi_\nu, \quad \dot{\Pi}_\nu = \frac{\partial}{\partial x} \frac{u_\nu(x)}{K_\nu(x)} \frac{\partial}{\partial x} \phi_\nu. \quad (15)$$

Let us first of all consider a non-interacting fermion gas, i.e.  $g = 0$ , without the optical lattice. This case is rather well-known (see for instance [14,15], where different approaches were used). The density of the gas in the LDA in terms of the local value of Fermi momentum is  $n(x) = 2k_F(x)/\pi\hbar$ . The internal energy of the gas is just the density of the kinetic energy (a.k.a. quantum pressure)  $E_0(n) = \hbar^2\pi^2 n^3(x)/24m$ . Substituting into Eq.(14) we find the ground state Thomas-Fermi density profile

$$n_{TF}(x) = n_0 \sqrt{1 - \frac{x^2}{R_{TF}^2}}, \quad (16)$$

for  $|x| < R_{TF}$ , and 0 otherwise. We have indicated with  $n_0 = (8\mu m/\hbar^2\pi^2)^{1/2}$  the central density of the trap and with  $R_{TF} = (2\mu/m\omega_x^2)^{1/2}$  the Thomas-Fermi size of the cloud. The chemical potential  $\mu = \hbar\omega_x N/2$  is fixed by the normalization requirement.

For an ideal gas the equations of motion Eq. (15) can be solved. Indeed in this case the Luttinger parameters have the values  $u_\nu = v_F$ ,  $K_c = 1$ , where the Fermi velocity  $v_F$  can be written in terms of the total density:  $v_F = \hbar\pi n/2m$ . Substituting in the equation of motion Eq. (15) and using the density profile Eq. (16), we can write down the equations for the density and spin-density fluctuations ( $\propto \partial_x \phi_\nu$ ):

$$-\omega_\nu^2 \rho_\nu = \omega_x^2 \frac{\partial}{\partial \tilde{x}} (1 - \tilde{x}^2)^{1/2} \frac{\partial}{\partial \tilde{x}} (1 - \tilde{x}^2)^{1/2} \rho_\nu, \quad (17)$$

where we defined the adimensional coordinate  $\tilde{x} = x/R_{TF}$ . The same equation has been derived in [15] using the mean field equations of Kolomeisky et al. [39]. The solution is given by

$$\sqrt{1 - \tilde{x}^2} \rho_{\nu n} = A_\nu \sin(\omega_{\nu n}/\omega_x \arccos \tilde{x}) \quad (18)$$

$$+ B_\nu \cos(\omega_{\nu n}/\omega_x \arccos \tilde{x}). \quad (19)$$

By analyzing the boundary conditions one can find the discrete spectrum of eigenfrequencies. The result:  $\omega_{\nu n} = \omega_x n$  [15] is the same both for the spin and the density modes. The first modes ( $n = 1$ ) has eigenfunction  $\rho_\nu \sim x/\sqrt{1 - \tilde{x}^2}$  and it corresponds to harmonic oscillations of the total density and the total spin (dipole and spin-dipole mode).

For weak interaction the correction to the ground state energy can be obtained by averaging the interparticle interaction over the free ground state:

$$E_0 = \hbar^2\pi^2 n^3/24m + gn^2/4. \quad (20)$$

Then, in the spirit of LDA using the Eq. (14), one finds that the density, up to the first order in  $\xi = g/\pi\hbar v_F(0)$ , uniformly decreases by  $\delta n(x) = -2gm/\hbar^2\pi^2$ . As expected the (repulsive) interaction reduces the density. This simple conclusion breaks down close to the edges of the cloud  $x = \pm R_{TF}$ , where the density goes to zero. Indeed we should require that  $g \ll \hbar^2 n(x)/m$ , i.e.,

$$(R_{TF} - x)/R_{TF} \lesssim (gm/\hbar^2 n_0)^2 \sim O(\xi^2). \quad (21)$$

The Luttinger parameters can be calculated substituting the local Fermi velocity  $v_F(x) = \pi\hbar(n_{TF}(x) + \delta n(x))/2m$  in the expressions obtained before for the homogeneous case with small interaction. In this way one finds

$$u_{c,s}(x) = \frac{\pi\hbar n_{TF}(x)}{2m} \left( 1 - \frac{A_{c,s}gm}{\pi^2\hbar^2 n_{TF}(x)} \right), \quad (22)$$

where  $A_s = 3$  and  $A_c = 1$  and to the same order  $K_c = 1 - gm/\pi^2\hbar^2 n_{TF}(x)$ .

In the strong interacting limit from the exact solution one has

$$E_0 = \hbar^2\pi^2 n^3/6m(1 - 4\hbar^2 n \ln(2)/mg). \quad (23)$$

The density profile for  $g \rightarrow \infty$  is still given by Eq. (16), but with  $n_0 \equiv n_\infty = (2m\mu/\pi^2\hbar^2)^{1/2}$ ,  $\mu = N\hbar\omega_x$  and  $R_{TF} = R_\infty = (2\mu/m\omega_x^2)^{1/2}$ . The first order correction to the density is  $\delta n(x) = 8\hbar^2 \ln(2)n_{TF}(x)^2/3mg$ . The spin velocity goes to zero as the interaction goes to infinity, the first order correction is simply

$$u_s(x) = \pi^3\hbar^3 n_{TF}(x)/3m^2g. \quad (24)$$

The density wave velocity and  $K_c$  have a non-zero value for infinite interaction and up to the first order correction one has  $K_c = 1/2(1 + 4\hbar^2 \ln(2)n_{TF}(x)/gm)$  and

$$u_c(x) = \frac{\pi\hbar n_{TF}(x)}{m} \left( 1 + \frac{4\hbar^2 \ln(2)n_{TF}(x)}{mg} \right).$$

On the same lines we can obtain the results in presence of the optical lattice. For the non-interacting case ( $g = 0 \Rightarrow U = 0$ ) the ground state energy of the gas is  $E_0 = 2Jn - 4J \sin(\pi nd/2)/\pi d$ . From Eq. (14) one has that the density of the cloud is

$$n_{TF} = \frac{2}{\pi d} \arccos \left[ 1 - \frac{\mu}{2J} (1 - \tilde{x}^2) \right] \quad (25)$$

for  $\tilde{x} = x/R_{TF} \leq 1$  and 0 elsewhere. From the previous expression we find the equations of motion for density and spin-density fluctuations in the ideal case

$$-\omega_\nu^2 \rho_\nu = \frac{m}{m^*} \omega_x^2 \frac{\partial}{\partial \tilde{x}} (1 - \tilde{x}^2)^{1/2} f_l(\tilde{x}) \frac{\partial}{\partial \tilde{x}} (1 - \tilde{x}^2)^{1/2} f_l(\tilde{x}) \rho_\nu, \quad (26)$$

where we defined  $f_l(\tilde{x}) = \sqrt{1 - \mu/4J(1 - \tilde{x}^2)}$ . For low density (i.e.  $f_l \rightarrow 1$ ) we have that the spectrum has the same structure as for the continuous case, but with a renormalized trapping frequency  $\omega_{\nu,n} = n\omega_x \sqrt{m/m^*}$ . This result is quite general and for instance the same correction was found in [40] for a Bose-Einstein condensate in 1D optical lattice.

The first order correction, calculated averaging the interaction  $\hbar\omega_c$  [Cœn term in Eq. (3)] over the ground state is

$\delta n = -Un_{TF}/2\pi J \sin(\pi n_{TF}d/2)$ . The density and spin-density wave velocities up to the first order are given by

$$u_{c(s)} = v_F(x) \left[ 1 - \frac{Ud}{2\hbar v_F} \left( \frac{n_{TF}d \cos(\pi n_{TF}d/2)}{\sin(\pi n_{TF}d/2)} \pm \frac{1}{\pi} \right) \right] \quad (27)$$

and to the same order the remaining parameter  $K_c$  has the same expression than in the homogeneous case but with a space dependent density given by the Eq. (25). In the strong interacting limit the energy density of the ground state can be written as

$$E_0 = 2J \left( n - \frac{\sin(\pi nd)}{\pi d} \right) + \frac{2J^2 \ln(2)}{Ud} \left[ \frac{nd}{\pi} \sin(2\pi nd) - (nd)^2 \right]. \quad (28)$$

The density profile for  $U \rightarrow \infty$  is given by the Eq.(25), divided by 2. The rest of the calculation proceeds on the same line as above. We will use these results later to discuss spin-charge separation.

We have shown how it is possible, under certain condition, to study as a Luttinger liquid an inhomogeneous 1D Fermi ultra-cold gas. We gave some analytical results in the two limiting case of weak and strong interaction. This will be used in the next section to study a possible way to see the equivalent of spin-charge separation in such system.

#### IV. SPIN-CHARGE SEPARATION IN TRAPPED COLD-GASES

##### A. Spectroscopy

The separation of the spin and the charge modes in the spectrum suggests experimental ways to observe the spin charge separation in harmonically trapped atoms [27]. We will propose two related experiments to observe the separation of the spin and the charge degrees of freedom. One relies on the possibility to excite easily some normal modes of the trapped gas and to measure their frequency. The other one consist in really testing the dynamics of total density and relative density wave packets. Before discussing the experiments it should be noticed that the application of the Hamiltonian Eq. (7) to finite systems deserves special attention. As shown in [32] the boundaries of a LL affect the physics of the lowest energy excitation  $\epsilon$  with  $\epsilon L/v_F \leq 1$ . Setting  $L \sim R_{TF}$  we find that the excitations with  $\omega_{\nu n} \sim \omega_x$  can have finite size corrections, which are not described by Hamiltonian Eq. (7) with the LDA values of the Luttinger parameters. Below we will consider  $\omega_{\nu n} \gg \omega_x$  (WKB approximation, see below) or small interactions. Hence we can neglect the finite size effect. Other finite size effect related to the TF

approximation will appear (logarithmic correction), but they are not related to Luttinger physics.

Assume we have the two different components of an interacting Fermi-system confined in external potentials, which act independently on each component. Then by modulating the two potentials simultaneously one induces oscillations of the total density. In turn, by slightly changing the confining potentials for the two components out of phase, only spin waves are excited. By observing the character and measuring the frequencies of the modes one can prove that the spin and the density modes are in fact the independent modes in the system and test the validity of the Luttinger description for such system. In FIG. 2 we show the numerical results of the level spacing for the spin and the charge modes for harmonically trapped atoms in the presence of optical lattice. Numerical results are based on the exact solution [37] together with the LDA.

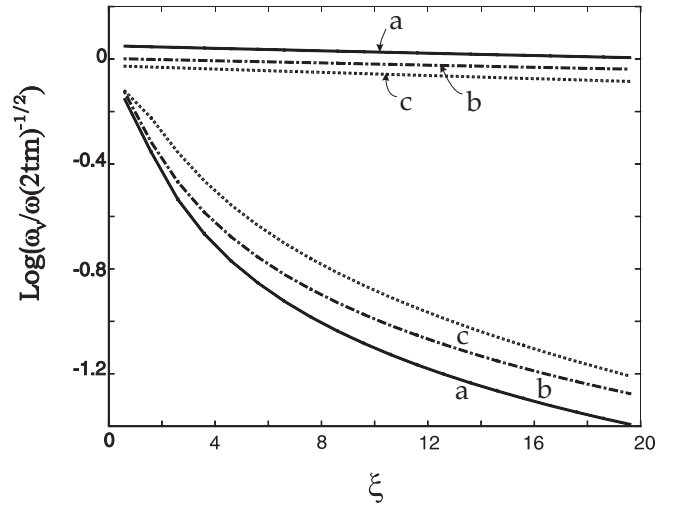


FIG. 2. Level spacing (in units of  $\omega_x$ ) of the spin (lower curves) and the charge (upper curves) modes for a harmonically trapped Fermi gas in the presence of an optical lattice (see the text). The level spacing is shown as a function of  $\xi = g/\pi\hbar v_F(0)$  and for different central densities: curves a:  $n(0) = 0.25/d$  (solid lines), curves b:  $n(0) = 0.42/d$  (dot-dashed lines), curves c:  $n(0) = 0.58/d$  (dotted lines).

For small interaction we can give an analytical expression to the energy correction of the lowest modes for a trapped gas using the balance equation Eq. (14). The frequency of the (density) dipole mode is unchanged as it has to be, while the correction to the spin dipole mode diverge logarithmically. The logarithmically divergent behaviour is due to the localization of the excitations of a free gas close to the border (see before and [15]). To handle the divergence we use as cut-off the condition Eq. (21) and we find

$$\delta\omega_s = -\omega_x \frac{3gm}{\pi^2\hbar^2 n_{TF}(0)} \log \frac{\pi\hbar^2 n_{TF}(0)}{2gm}.$$

The energy shift is negative and can be measured by comparing the density and the spin oscillation of the atomic cloud. The logarithm in the previous expression tells us that in harmonic traps the requirement to apply the perturbation theory in  $\xi$  is stronger than in the homogeneous case, namely it should be  $\xi \log(1/\xi) \ll 1$ . We can say that the LDA is valid with logarithmic accuracy.

The same behaviour for the first order correction to the spin-dipole frequency can be obtained in this regime also using sum rules. This was done for a 3D system in [16]. Generally speaking, it is possible to show, that an upper bound to the energy  $\hbar\omega_F$  of the lowest state excited by an operator  $F$  is given by the ratio  $\hbar^2\omega_F^2 = m_3/m_1$ , where  $m_k$  is the  $k$ th moment of the dynamic structure factor. One can write the two moments as

$$\begin{aligned} m_1 &= \frac{1}{2} \langle 0 | [F^\dagger, [H, F]] | 0 \rangle, \\ m_3 &= \frac{1}{2} \langle 0 | [[F^\dagger, H], [H, [H, F]]] | 0 \rangle, \end{aligned} \quad (29)$$

where  $H$  is the Hamiltonian and  $|0\rangle$  the ground state of the system. We are interested in the excitations due to the spin dipole operator  $S = \sum_{i\uparrow} x_i - \sum_{i\downarrow} x_i$  and the Hamiltonian being Eq. (1). In the limit of small interaction one has

$$\omega_s = \omega_x - \frac{g}{mN\omega_x} \int dx |\partial_x n(x)|^2. \quad (30)$$

Substituting for the ground state density  $n$ , the density profile Eq. (16) and using the cut-off Eq. (21), one finds, to logarithmic precision

$$\delta\omega_s = -\omega_x \frac{16}{\pi} \frac{gm}{\pi^2 \hbar^2 n_{TF}(0)} \log\left(\frac{\hbar^2 n_{TF}(0)}{gm}\right). \quad (31)$$

The Eq. (30) has nothing to do with Luttinger model and, hence, provides an alternatives derivation for the spin excitation frequency. The fact that the results agree with each other confirms both the applicability of the Luttinger model and the LDA.

We will introduce now a WKB calculation of the energy excitation, which we will allow us to calculate the energy shift of the spin excitations in the strong interacting regime. Suppose that the energy levels of a quantum system are characterized by a certain quantum number  $n$ . For sufficiently high quantum number the eigenfrequencies can be obtained from the WKB (Bohr-Sommerfeld) quantization condition,

$$\int_{-x_0}^{x_0} p(x) dx = \hbar\pi(n + \alpha), \quad (32)$$

where  $p(x)$  is the WKB momentum corresponding to a given energy,  $x_0$  is the classical turning point and the constant  $\alpha$  is the so called Maslov index. The accuracy of the WKB spectrum estimation is  $\sim 1/(\pi n)^2$ . In our case we can fix  $\alpha$  by comparing the WKB results and

the exact solutions of Eqs. (15) for a weakly interacting gas. The spectrum of the excitations we are considering is linear,  $\epsilon = u_{\rho,s}(x)p(x)$ , with the velocities given by the Eq. (22). Using the Eq. (32) we obtain the same sort of logarithmically diverging integrals as those we obtained in the perturbative calculation above. Once again to regularize such diverging behavior we use the condition (21) and we find

$$\epsilon_{\rho,s} = \hbar\omega(n+1) \left( 1 - \frac{2gmA_{\rho,s}}{\pi^2 \hbar^2 n_{TF}(0)} \log\left(\frac{\hbar^2 n_{TF}(0)\pi}{mg}\right) \right). \quad (33)$$

We find once more that the interaction dependent splitting of the eigenenergies is different for charge and spin.

In the strong interaction limit, after integrating Eq. (32), we find that once again the density dipole mode is unchanged. The leading term of the energy of the spin dipole mode, which is zero in the limit  $\xi \rightarrow \infty$ , is given by

$$\epsilon_s(p) = \frac{\hbar^4 \pi^3 n^2(x) p(x)}{3m^2 g}.$$

Substituting  $p(x)$  in the integral equality (32) and dealing with the logarithmic divergence as usual, we find, that

$$\epsilon_{ns} = \hbar\omega(n + \alpha) \frac{\hbar^5 \pi^3 n_\infty}{3gm \log(gm/\hbar^2 n_\infty)}, \quad (34)$$

where  $\alpha \sim 1$ . In the limit of the strong interaction the spin level spacing decreases a lot and is much smaller than that between the density waves ( $\omega_x$ ), as is clearly shown in FIG.2.

## B. Wavepacket dynamics

The difference between the spin and the charge velocities allows to have spin and charge wave packets moving at different velocities. This can be seen in a sufficiently long trap. Localized wave packets of the spin and density excitations can be generated by sufficiently localized optical potentials of dimension  $\ell$  such that  $R \gg \ell \gg k_F^{-1}$ , where  $R$  is the size of the atom cloud and the Fermi momentum  $k_F$  is essentially the interparticle distance FIG. 3.



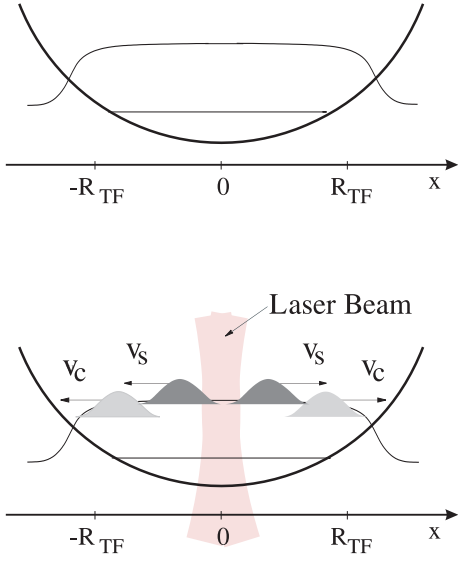


FIG. 3. Two component quasi-1D Fermi gas in a harmonic trap. A short laser pulse focused near the center of the trap can excites density and spin density wave packet.

The procedure we have in mind is analogous to the MIT setup originally used to study propagation of sound waves in elongated condensates [41]. The motion of these wave packets is readily detected by internal state dependent laser probes. In FIG. 4 we show a simulate wave packets motion for the states  $|F = 1/2, M_F = \pm 1/2\rangle$  of  $^6\text{Li}$  with interaction parameter  $\xi = 1$ . This figure was obtained solving numerically Eq. (15). The scattering length of these two states is magnetically tunable and recently the tunability has been experimentally explored up to high magnetic field (see [42] references therein). The scattering length is positive for applied magnetic field  $B \geq 550\text{G}$  and presents a Feshbach resonance around  $800\text{G}$ . We can, for instance, consider small scattering length of the order of  $a = 20\text{\AA}$ . In order to have  $\xi \sim 1$ , one can use  $N = 500$  particles at trap frequencies  $\omega_x = 1\text{Hz}$  and  $\omega_\perp = 250\text{kHz}$ . On the other hand near the Feshbach resonance the scattering length can be larger by orders magnitude, with respect to the previous value. For  $a$  just ten times bigger than the previous one it is possible to have  $N = 1000$  atoms at a trap frequency  $\omega_\perp = 100\text{kHz}$ , leaving unchanged  $\omega_x$ .

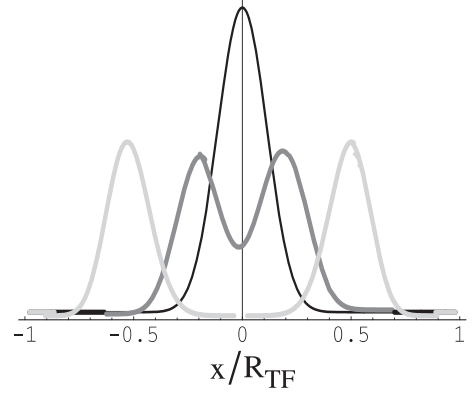


FIG. 4. Wave packet dynamics for different times as function of position (in units of the Thomas Fermi radius  $R_{TF}$ ): spin-charge separation manifests itself in a spatial separation of the spin (solid line) and density (dashed line) wave packets (shown at half a trap oscillation period  $\omega t = \pi/2$ ), which can be probed by a second short laser pulse at a later time. The parameters correspond to  $N = 10^3$   $^6\text{Li}$  atoms in a trap with  $\omega = 1\text{Hz}$  (with coupling parameter  $\xi = 1$ , see text).

An optical potential created by a far detuned laser light and acting in the same way on both the components amounts to an external potential  $\sim \partial_x \phi_c$  and hence acts only on the density waves. An optical potential generated by a laser tuned, e.g., between fine structure levels of excited Alkali states acts on the ground state spin in a way equivalent to an external magnetic field interacting with the spin density  $\sim \partial_x \phi_s$  and thus act solely on the spin waves. One can also act at the same time coherently on both the charge and the spin waves. We analyze in some detail this case (the same can be done for the previous cases). The idea is sketched in FIG. (5). We need to couple only one out of the two (“long-living”) hyperfine levels identified by  $|\sigma\rangle = |\uparrow\rangle, |\downarrow\rangle$ , with another (in principle “short-living”) state of the atom,  $|e\rangle$ . Let  $\Omega$  be the Rabi frequency which couples for instance  $|\uparrow\rangle$  and  $|e\rangle$ ,  $\delta$  be the detuning, i.e.  $\delta = \omega_L - \omega_{\uparrow e}$ , where  $\omega_L$  is the laser frequency and  $\hbar\omega_{\uparrow e}$  is the energy separation between  $|\uparrow\rangle$  and  $|e\rangle$ , and  $\gamma$  be the spontaneous emission rate of the state  $|e\rangle$ . In the regime of large detuning,  $\delta \gg \Omega, \gamma$  the effect of the coupling is to introduce an energy shift  $\Omega_\uparrow = |\Omega|^2/\delta$  and a decay rate  $\Gamma = |\Omega/\delta|^2\gamma$ . Thus, for not too long times ( $\Gamma t \ll 1$ ), we have the Hamiltonian density  $\mathcal{H}_{\text{ext}}(x) = \Omega_\uparrow(x)\psi_\uparrow^\dagger(x)\psi_\uparrow(x)$  acting on the gas. Eventually, if  $\Omega_\uparrow(x)$  (i.e. the laser profile) is slowly varying in space, to avoid scattering processes,  $q \simeq 2k_F$ , in terms of charge and spin fields one can write

$$H_{\text{ext}} = \int \frac{dx}{\sqrt{2\pi}} \Omega_\uparrow(x) (\partial_x \phi_c(x) + \partial_x \phi_s(x)). \quad (35)$$

Clearly the previous Hamiltonian acts as localized (with the above remark) independent perturbations on both charge and spin densities.



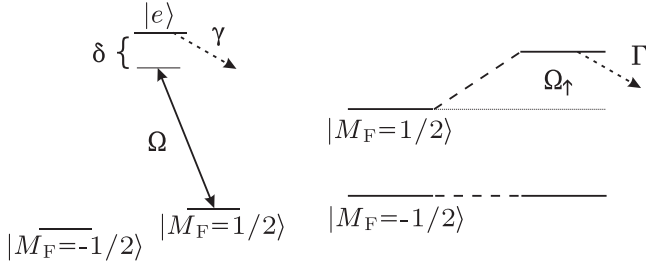


FIG. 5. Internal level scheme of the atoms of the gas. The state  $|\uparrow\rangle = |M_F = 1/2\rangle$  is coupled via a laser to a (in general unstable) state  $|e\rangle$  (left). The coupling being given by the Rabi frequency  $\Omega$ . On the right, we sketched, the effective action of the laser for large detuning  $\delta \gg \Omega, \gamma$  (see the text).

## V. RELAXATION PHENOMENA

As stated before, the Hamiltonian Eq. (7) describes properly the fluctuations of a 1D system only in the long wave  $q/k_F \ll 1$ , low energy regime. Higher order terms originating from, for example, the non-linearity of the fermionic spectrum appear which can mix the excitations. The first corrections are third power in  $\Pi_\nu$  and  $\phi_\nu$ , which lead to scattering of the excitations. This is precisely the case in the simple model described by Eq. (4). Introducing once again the operators  $c_{r,k,\sigma}$ ,  $r = R, L$  and  $\sigma = \uparrow, \downarrow$  for the left- and right-movers in the momentum space, but this time keeping the terms up to the second order in  $k - k_F$ , one has for the kinetic part

$$H_0 = \sum_{r,k,\sigma} (\partial_k \epsilon)_{k_F} (a_r k - k_F) c_{r,k,\sigma}^\dagger c_{r,k,\sigma} \quad (36)$$

$$+ \frac{(\partial_k^2 \epsilon)_{k_F}}{2} (a_r k - k_F)^2 c_{r,k,\sigma}^\dagger c_{a,k,\sigma}, \quad (37)$$

where  $a_{R(L)} = +(-)$ . The quadratic term in real space reads  $(\partial_k^2 \epsilon)_{k_F} \sum_{r,\sigma} \int dx : \partial_x \psi_{r,\sigma}^\dagger(x) \partial_x \psi_{r,\sigma}(x)$  and using the bosonization identity (see Appendix Eq. (A1)) one obtains  $4\sqrt{\pi}(\partial_k^2 \epsilon)_{k_F} \sum_{r,\sigma} \int dx (\partial_x \phi_{a,\sigma})^3$ . Introducing the charge and spin boson fields Eq. (6), we find that we have to add the following third order term to the Luttinger model density Hamiltonian

$$\begin{aligned} \mathcal{H}_{\text{int}} = & \frac{(\partial_k^2 \epsilon)_{k_F}}{\sqrt{2\pi}} \{ (\partial_x \phi_c)^3 + 3\Pi_c (\partial_x \phi_s \Pi_s + \Pi_s \partial_x \phi_s) \\ & + \partial_x \phi_c \Pi_c^2 + \Pi_c \partial_x \phi_c \Pi_c + \Pi_c^2 \partial_x \phi_c \\ & + 3\partial_x \phi_c [(\partial_x \phi_s)^2 + \Pi_s^2] \}. \end{aligned} \quad (38)$$

To study the relaxation phenomena we will consider, for simplicity, a single component gas (polarized system). In the previous Hamiltonian we are left only with the term  $\mathcal{H}_{\text{int}} = \hbar^2 (\partial_x \phi)^3 / \sqrt{2\pi} m$ .

According to Haldane [19], the generic term due to the non-linearity of the spectrum is given by  $V_{\text{int}} = \gamma \hbar^2 / m \int (\partial_x \phi)^3$ , where  $\gamma \sim 1$ .

We calculated the damping of the oscillations by the Fermi's Golden rule and the second quantized representation for the phonon fields Eqs. (A2) and (A3). We considered the decay rate of an excitation with momentum  $q$  and energy  $\omega_q$ . In the limiting case  $k_B T \ll \hbar \omega_q$ , a straightforward calculation gives

$$\Gamma_{T=0} \sim \hbar^2 q^4 L / m^2 u. \quad (39)$$

This is the decay rate in a process where a particle with the energy  $\hbar \omega_q$  decays into a pair of particles with  $\omega_1, \omega_2 < \omega_q$ . Note that  $\omega_q \gg \omega_x$ , otherwise there are no final states for such decay instability (i.e. the lowest excitations are very stable at very low temperatures). In the case of  $k_B T \gg \hbar \omega_q$  (but still  $T \ll T_F$ ) we find

$$\Gamma_T \sim q^2 L k_B^2 T^2 / m^2 u^3 \quad (40)$$

which corresponds to Landau damping, i.e., the contribution of a process in which the damping occurs by scattering a high-frequency excitation with  $\hbar \omega'_q \sim k_B T$ . Note that both results contain the size of the sample  $L$ . For sufficiently small temperatures (or high number of particles) the excitations are only weakly damped. Indeed, for the lowest excitations  $L m u / \hbar \sim N$ ,  $\omega_q \sim \omega \sim \epsilon_F / \hbar N$ , thus substituting the previous expression for the decay rate we have

$$\Gamma / \omega_q \sim (k_B T / \epsilon_F)^2 \ll 1. \quad (41)$$

In conclusion for sufficiently low temperature or large enough number of particles the excitations are only weakly damped by the third order correction to the otherwise quadratic Luttinger Hamiltonian. This means that in a degenerate quasi-1D Fermi gas the excitations can be observed for several trap periods.

## VI. SPINLESS LL WITH BOSONS

In this section we will show how to realize a spinless LL Hamiltonian using “hard core” bosons in an optical lattice. The on site interaction of the bosons can be written as  $U(n_i - 1)n_i$ . Hence, in the limit of  $g \rightarrow \infty$  ( $U \rightarrow \infty$ ) only the states  $n_i = 0, 1$  are allowed. We parameterize them as  $\alpha = \pm$  states of a particle with the spin  $1/2$ . In terms of spin operators (Pauli matrices) the lattice (Hubbard) Hamiltonian can be put in the form

$$H = J \sum_i (\sigma_i^+ \sigma_{i+1}^- + h.c.) + V \sum_i (\sigma_i^z + 1)(\sigma_{i+1}^z + 1). \quad (42)$$

Where  $V$  is due to the off-site interaction, which is present in the starting bosonic Hamiltonian. Usually one has  $V \ll U$ . In the easiest case, if  $d$  is the lattice period and  $w(x)$  the Wannier function, one has

$$U = g \int dx |w(x)|^4 \quad (43)$$

$$V = g \int dx |w(x)w(x+d)|^2. \quad (44)$$

The Hamiltonian Eq. (42) can be brought to a fermionic form with the help of Jordan-Wigner transformation:

$$\begin{aligned} \sigma_i^+ &= a_i^\dagger \prod_{j<i} \sigma_j^z, \\ \sigma_i^- &= \left( \prod_{j<i} \sigma_j^z \right) a_i, \\ \sigma_i^z &= 2a_i^\dagger a_i - 1, \end{aligned} \quad (45)$$

where the  $a_i$ 's are fermionic operator. Substituting these expressions into the spin-chain Hamiltonian we get:

$$H = J \sum_i (a_i^\dagger a_{i+1} + h.c.) + 4V \sum_i a_i^\dagger a_i a_{i+1}^\dagger a_{i+1}. \quad (46)$$

Thus we have obtained a 1D system of interacting spin polarized fermions starting with a system of bosons. Bosonizing the JW fermions we arrive at a LL Hamiltonian similar to Eq. (7).

We should also mention that, recently, was shown that it is possible to study typical fermionic correlation phenomena using 2-level bosonic atoms trapped in optical lattices [43].

## VII. CONCLUSION

In conclusion, we have described 1D (Fermi) gases under different trapping configurations as Luttinger liquids by using the available exact solutions. We have shown how it is possible to gain some physical insights with this approach also in the non-interacting case (strictly speaking we cannot talk in this case of Luttinger liquids). In addition, we have proposed some experiments to test the validity of this approach and to see in a clear way the so-called "spin-charge separation".

We emphasize that a series of approximations has been made. We considered our system as an infinite one, or better finite with periodic boundary conditions, to extract the Luttinger parameters. We commented on that for what concern an actual experiment. The finite-size correction can be, in any case with some effort, taken into account. The analysis of the inhomogeneous case was carried out within a local density approximation. For what we gave the range of validity.

In summary, cold quantum degenerate Fermi gases are a promising avenue to observe Luttinger physics in a novel system in the laboratory.

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## APPENDIX A: BOSONIZATION IDENTITIES

It is possible to represent the single fermion field operators  $\psi_{r,\sigma}$  ( $r = R, L$  and  $\sigma = \uparrow, \downarrow$ ) in terms of the boson field  $\phi_\nu$  and  $\Pi_\nu$  ( $\nu = c, s$ ) (see for instance [20,34,22,31]) by the bosonization identity

$$\psi_{r,\sigma} = \lim_{\alpha \rightarrow 0} \frac{F_{r,\sigma}}{(2\pi\alpha)^{1/2}} e^{-a_{r,i} 2\sqrt{\pi}\phi_{R(L),\sigma}}, \quad (A1)$$

where  $a_{R(L)} = +(-)$ . The fields  $\phi_{r,\sigma}$  are recovered from Eq. (6). The (hermitian) operators  $F_{r,\sigma}$  are called Klein factors and obey the Clifford algebra  $\{F_{r,\sigma}, F_{r',\sigma'}\} = 2\delta_{r,r'}\delta_{\sigma,\sigma'}$ . We remind that the Luttinger model is based on an infinite filled Dirac sea. The last correspondence must be handled with utmost care when a product of two (or more) fermion operators with the same argument are considered. In such cases one must implement the normal ordering prescription, i.e.,  $:A := A - \langle A \rangle_{\text{vac}}$ . For instance the right way to compute the densities is by using the limit prescription:  $\rho(x) = \lim_{\varepsilon \rightarrow 0} : \psi^\dagger(x)\psi(x+\varepsilon) :$  [31].

It is useful to write the boson fields in terms of boson creation and annihilation operators,  $b_{\nu,k}^\dagger$  and  $b_{\nu,k}$ ,  $\nu = c, s$

$$\begin{aligned} \phi_\nu(x) &= \sqrt{\frac{K_\nu}{2L}} \sum_k \frac{1}{\sqrt{|k|}} \left( b_{\nu,k} e^{ikx} + b_{\nu,k}^\dagger e^{-ikx} \right) \\ &\quad + \frac{\sqrt{\pi}x}{L} \hat{N}_\nu \end{aligned} \quad (A2)$$

$$\begin{aligned} \Pi_\nu(x) &= i\sqrt{\frac{1}{2LK_\nu}} \sum_k \sqrt{|k|} \left( b_{\nu,k} e^{ikx} - b_{\nu,k}^\dagger e^{-ikx} \right) \\ &\quad - \frac{\sqrt{\pi}}{L} \hat{J}_\nu. \end{aligned} \quad (A3)$$

The eigenvalues of the operators  $\hat{N}_\nu$ 's are the number of particles added to the ground state ( $\nu = c$ ) and the "magnetization" ( $N_\uparrow - N_\downarrow$ ) of the system with respect to the ground state ( $\nu = s$ ). The operators  $\hat{J}_\nu$  represent the respective currents. In the thermodynamic limit ( $L \rightarrow \infty$ ) or for excitations which do not change neither the number of particles and the currents these operators can be neglected. Note that such "topological" excitations can be recognized also in the exact solutions of the models we discussed in this paper.

## 1. Backward and umklapp scattering

Using the previous identities it is possible to express also the backward scattering in terms of the bosonic fields  $\phi_\nu$  and  $\Pi_\nu$ . In terms of fermionic operators the backward scattering is

$$H_b = g_1 \sum_{k_1, k_2, \sigma} c_{R, \sigma}^\dagger(k_1) c_{L, \sigma}(k_1 - q) c_{L, -\sigma}^\dagger(k_2) c_{R, -\sigma}(k_2 + q). \quad (\text{A4})$$

Expressing the previous term in the real space and substituting the fermionic fields with the Eq. (A1) one gets

$$\mathcal{H}_b(x) = 4g_{1\perp} F_{R\uparrow} F_{R\downarrow} F_{L\uparrow} F_{L\downarrow} \cos(\sqrt{8}\phi_s). \quad (\text{A5})$$

The product of Klein factors can in this case substitute (with a kind of gauge choice) with +1 (see [20,31,34] for a comprehensive discussion). From the obtained expression is quite clear that, when  $g_{1\perp}$  renormalizes to strong coupling, one has a gap for the spin excitations [44]. This is the case, e.g., for  $g_{1\perp} < 0$  and  $K_s = 1$ . When the lattice is present also an umklapp term can be present. The “bosonized” form of such term is essentially like Eq. (A5), which  $\phi_c$ , instead of  $\phi_s$  in the argument of the cosine function. But in this case a gap can be opened only at half filling [45].

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